Introduction to Mathematical Quantum Theory

Solution to the Exercises

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Exercise 1

Let $\mathcal{H} := L^2(\mathbb{R})$ and $P := -i\partial_x$ the momentum operator defined on the domain $\mathcal{D}(P) := H^1(\mathbb{R})$ as $P\psi(x) = -i\frac{\partial\psi}{\partial x}(x)$. Consider for any $\lambda \in \mathbb{R}$ the bounded operator T_λ defined for any $\psi \in \mathcal{H}$ as $T_\lambda\psi(x) = \psi(x-\lambda)$.

Prove that $\{T_\lambda\}_{\lambda\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group and that

$$T_\lambda = e^{i\lambda P} = e^{\lambda\partial_x}. \tag{1}$$

Proof. By definition it follows that if $\lambda = 0$ then for any $\psi \in \mathcal{H}$ we get $T_0\psi(x) = \psi(x)$ and therefore $T_0 = \mathrm{id}$. On the other hand, let $\lambda,\ \mu \in \mathbb{R}$; then for any $\psi \in \mathbb{H}$ we get $T_\lambda T_\mu\psi(x) = T_\mu\psi(x-\lambda) = \psi(x-\lambda-\mu) = T_{\lambda+\mu}(x)$. Consider now $\lambda \in \mathbb{R}$, $\psi,\ \varphi \in \mathcal{H}$; to prove that T_λ is a unitary operator we compute T_λ^* to get

$$\langle\psi, T_\lambda^*\varphi\rangle = \langle T_\lambda\psi, \varphi\rangle = \int_{\mathbb{R}} \overline{\psi(x-\lambda)}\varphi(x)\,dx = \int_{\mathbb{R}} \overline{\psi(x)}\varphi(x+\lambda)\,dx = \langle\psi, T_{-\lambda}\varphi\rangle,$$

and as a consequence $T_\lambda^* = T_{-\lambda}$; therefore we get $T_\lambda T_\lambda^* = T_\lambda T_{-\lambda} = T_0 = \mathrm{id}$, $T_\lambda^* T_\lambda = T_{-\lambda}T_\lambda = T_0 = \mathrm{id}$ and for any $\lambda \in \mathbb{R}$, T_λ is unitary.

Consider now A the infinitesimal generator of $\{T_\lambda\}_{\lambda\in\mathbb{R}}$. Suppose $\psi \in \mathcal{H}$; recall that if $\mathcal{F}$ represent the Fourier transform, we get

$$\mathcal{F}T_\lambda\psi(k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}\psi(x-\lambda)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ik(x+\lambda)}\psi(x)\,dx = e^{-i\lambda k}\widehat{\psi}(k).$$

Suppose now that $\psi \in H^1(\mathbb{R})$; then we get for any λ

$$\mathcal{F}\left(\frac{T_\lambda - \mathrm{id}}{\lambda}\psi\right)(k) = \frac{e^{-i\lambda k}-1}{\lambda}\widehat{\psi}(k).$$

Given that $\mathcal{F}P\psi(k) = k\widehat{\psi}(k)$, we get

$$\left\|\frac{T_\lambda - \mathrm{id}}{\lambda}\psi - iP\psi\right\|_2^2 = \left\|\frac{e^{-i\lambda k}-1}{\lambda}\widehat{\psi} - ik\widehat{\psi}\right\|_2^2 = \int_{\mathbb{R}}\left|\frac{e^{-i\lambda k}-1}{\lambda} - ik\right|^2 |\psi(k)|^2\,dk$$

Given that $\left|\frac{e^{-i\lambda k}-1}{\lambda}\right|^2 \leqslant |k|^2$, and $|k|^2\left|\widehat{\psi}(k)\right|^2 \in L^1(\mathbb{R})$, we can conclude

$$\lim_{\lambda\to 0}\left\|\frac{T_\lambda - \mathrm{id}}{\lambda}\psi - iP\psi\right\|_2^2 = \int_{\mathbb{R}}\lim_{\lambda\to 0}\left|\frac{e^{-i\lambda k}-1}{\lambda} - ik\right|^2 |\psi(k)|^2\,dk = 0.$$

So clearly $H^1(\mathbb{R}) \subseteq \mathcal{D}(A)$ and A is an extension of P. Now, given that as a consequence we have that P^* is an extension of A^* and that both A and P are self-adjoint, we get that $A = P$; therefore by Stone theorem $e^{\lambda\partial_x} = e^{i\lambda P} = T_\lambda$.

□

Exercise 2

Let $\mathcal{H}$ be an Hilbert space, A a symmetric operator and $\mu > 0$ a positive real number. Prove that the following are equivalent.

a A is self-adjoint.

b $\operatorname{Ran}(A + i\mu\,\mathrm{id}) = \operatorname{Ran}(A - i\mu\,\mathrm{id}) = \mathcal{H}$.

Proof. In class we proved that if A is symmetric, $A = A^*$ if and only if $\operatorname{Ran}(A \pm i\,\mathrm{id}) = \mathcal{H}$. Now, A is symmetric if and only if $\frac{1}{\mu}A$ is, and given that $A + i\mu\,\mathrm{id} = \mu\left(\frac{1}{\mu}A + i\,\mathrm{id}\right)$, $\operatorname{Ran}(A + i\mu\,\mathrm{id}) = \operatorname{Ran}\left(\frac{1}{\mu}A + i\,\mathrm{id}\right)$, the result follows from the result proven in class.

□

Exercise 3

Let $\mathcal{H}$ be an Hilbert space. Let $U \in \mathcal{B}(\mathcal{H})$. Prove that U is unitary if and only if there exist a self-adjoint operator A on $\mathcal{H}$ such that $U = e^{iA}$.

Proof. Recall that from functional calculus for self-adjoint operators we have $f(A)^* = \overline{f}(A)$. Then, given that $\overline{e^{ix}} = e^{-ix}$, we get $\left(e^{iA}\right)^* = e^{-iA}$. As a consequence we get $e^{iA}\left(e^{iA}\right)^* = e^{iA}e^{-iA} = \mathrm{id} = e^{-iA}e^{iA} = \left(e^{iA}\right)^* e^{iA}$, and therefore $U = e^{iA}$ is a unitary operator.

Suppose now U is unitary. Then we get $\sigma(U) \subseteq \overline{B_1(0)}$ because $\|U\| = 1$. On the other hand we get that if $\lambda \in B_1(0)$, then $U - \lambda\,\mathrm{id} = U(\mathrm{id} - \lambda U^*) = (\mathrm{id} - \lambda U^*)\,U$, and given that $\|\lambda U^*\| = |\lambda| < 1$ and U is unitary, then $U - \lambda\,\mathrm{id}$ is invertible and $\lambda \notin \sigma(U)$. Therefore $\sigma(U) \subseteq \mathbb{S}_1$ (where $\mathbb{S}_1 = \{\psi \in \mathcal{H} |\ \|\psi\| = 1\}$).

Now, the map x^t is bounded from $\sigma(U) \to \mathbb{C}$ for any $t \in \mathbb{R}$. Define then $U(t) := U^t$, defined through the functional calculus for normal operators. By construction we have that $U(t)$ is a strongly continuous one-parameter unitary group, so let A be the self-adjoint infinitesimal generator. As a consequence of Stone theorem, we get that $U(t) = e^{itA}$, and therefore, $U = U(1) = e^{iA}$.

□

Exercise 4

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \mathrm{id}, \tag{2}$$
$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{3}$$

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^* A_+ + A_-^* A_-\right) + \zeta\left(A_+^* A_-^* + A_+ A_-\right). \tag{4}$$

- **a** Prove that H is self-adjoint.
- **b** Prove that there exist operators $C_\pm$ and numbers $\alpha,\ \beta \in \mathbb{R}$ such that

$$\left[C_\pm, C_\pm^*\right] = \mathrm{id}, \tag{5}$$
$$\left[C_+, C_-\right] = \left[C_+, C_-^*\right] = 0, \tag{6}$$
$$H = \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta. \tag{7}$$

Hint: Define

$$C_\pm := \gamma_\pm A_\pm + \xi_\pm A_\mp^* \tag{8}$$

for some $\gamma_\pm,\ \xi_\pm \in \mathbb{R}$. *Use* (5) *and* (6) *to deduce that* $\gamma_+ = \gamma_-$, $\xi_+ = \xi_-$ *and that* $\gamma_\pm^2 - \xi_\pm^2 = 1$. *Calculate* $C_\pm^* C_\pm$ *and deduce* (7).

Proof. To prove **a**, given that $A_\pm$ are bounded operators, first notice that $\left(A_\pm^* A_\pm\right)^* = A_\pm^* A_\pm^{**} = A_\pm^* A_\pm$. On the other hand $\left(A_+^* A_-^*\right)^* = A_-^{**} A_+^{**} = A_- A_+ = A_+ A_-$, and therefore $\left(A_+^* A_-^* + A_+ A_-\right)^* = A_+^* A_-^* + A_+ A_-$. As a consequence H is self-adjoint.

To prove **b** , consider $C_\pm$ defined as (8). Using (5) we get

$$\begin{aligned}\mathrm{id} = \left[C_\pm, C_\pm^*\right] &= \left[\gamma_\pm A_\pm + \xi_\pm A_\mp^*, \gamma_\pm A_\pm^* + \xi_\pm A_\mp\right] \\ &= \gamma_\pm^2\left[A_\pm, A_\pm^*\right] + \xi_\pm^2\left[A_\mp^*, A_\mp\right] = \left(\gamma_\pm^2 - \xi_\pm^2\right)\mathrm{id}\,.\end{aligned}$$

Now, given that the function sinh is bijective, let $\theta_\pm \in \mathbb{R}$ such that $\xi_\pm = \sinh(\theta_\pm)$. Then $\gamma_\pm^2 = 1 + \xi_\pm^2 = 1 + \sinh^2(\theta_\pm) = \cosh(\theta_\pm)$. Using (6) we then get

$$\begin{aligned}0 = \left[C_+, C_-\right] &= \left[\gamma_+ A_+ + \xi_+ A_-^*, \gamma_- A_- + \xi_- A_+^*\right] \\ &= \gamma_+ \xi_- \left[A_+, A_+^*\right] + \xi_+ \gamma_- \left[A_-^*, A_-\right] = \left(\gamma_+ \xi_- - \xi_+ \gamma_-\right)\mathrm{id} \\ &= \left(\cosh(\theta_+)\sinh(\theta_-) - \cosh(\theta_-)\sinh(\theta_+)\right)\mathrm{id} = \sinh(\theta_- - \theta_+)\,\mathrm{id}\,.\end{aligned}$$

From the fact that $\sinh^{-1}(0) = 0$, we get that $\theta_+ = \theta_- = \theta$. We then got that $C_\pm = \cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*$, and we now consider $C_\pm^* C_\pm$:

$$\begin{aligned}C_\pm^* C_\pm &= \left(\cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*\right)^* \left(\cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*\right) \\ &= \left(\cosh(\theta) A_\pm^* + \sinh(\theta) A_\mp\right)\left(\cosh(\theta) A_\pm + \sinh(\theta) A_\mp^*\right) \\ &= \cosh^2(\theta) A_\pm^* A_\pm + \sinh^2(\theta) A_\mp A_\mp^* + \sinh(\theta)\cosh(\theta)\left(A_+^* A_-^* + A_+ A_-\right) \\ &= \cosh^2(\theta) A_\pm^* A_\pm + \sinh^2(\theta) A_\mp^* A_\mp + \sinh^2(\theta)\left[A_\mp, A_\mp^*\right] \\ &\qquad + \sinh(\theta)\cosh(\theta)\left(A_+^* A_-^* + A_+ A_-\right) \\ &= \cosh^2(\theta) A_\pm^* A_\pm + \sinh^2(\theta) A_\mp^* A_\mp \\ &\qquad + \sinh(\theta)\cosh(\theta)\left(A_+^* A_-^* + A_+ A_-\right) + \sinh^2(\theta)\,.\end{aligned}$$

Using the fact that $\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)$ and $\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(2\theta)$, as a consequence we get

$$\begin{aligned} C_+^* C_+ + C_-^* C_- &= \cosh^2(\theta)\, A_+^* A_+ + \sinh^2(\theta)\, A_-^* A_- \\ &\quad + \sinh(\theta)\cosh(\theta)\left(A_+^* A_-^* + A_+ A_-\right) + \sinh^2(\theta) \\ &\quad + \cosh^2(\theta)\, A_-^* A_- + \sinh^2(\theta)\, A_+^* A_+ \\ &\quad + \sinh(\theta)\cosh(\theta)\left(A_+^* A_-^* + A_+ A_-\right) + \sinh^2(\theta) \\ &= \cosh(2\theta)\left(A_+^* A_+ + A_-^* A_-\right) + \sinh(2\theta)\left(A_+^* A_-^* + A_+ A_-\right) \\ &\quad + 2\sinh^2(\theta)\,. \end{aligned}$$

Now notice that

$$\left(\frac{\eta}{\sqrt{\eta^2-\zeta^2}}\right)^2 - \left(\frac{\zeta}{\sqrt{\eta^2-\zeta^2}}\right)^2 = 1.$$

As a consequence, there is θ such that $\cosh(2\theta) = \frac{\eta}{\sqrt{\eta^2-\zeta^2}}$ and $\sinh(2\theta) = \frac{\zeta}{\sqrt{\eta^2-\zeta^2}}$, and as a consequence

$$\begin{aligned} H &= \eta\left(A_+^* A_+ + A_-^* A_-\right) + \zeta\left(A_+^* A_-^* + A_+ A_-\right) \\ &= \sqrt{\eta^2-\zeta^2}\left[\cosh(2\theta)\left(A_+^* A_+ + A_-^* A_-\right) + \sinh(2\theta)\left(A_+^* A_-^* + A_+ A_-\right)\right] \\ &= \sqrt{\eta^2-\zeta^2}\left[C_+^* C_+ + C_-^* C_- - 2\sinh^2(\theta)\right], \end{aligned}$$

so $\alpha = \sqrt{\eta^2-\zeta^2}$. Now, we have

$$\begin{aligned} -2\sqrt{\eta^2-\zeta^2}\sinh^2(\theta) &= \sqrt{\eta^2-\zeta^2}\left(1-\cosh(2\theta)\right) = \sqrt{\eta^2-\zeta^2}\left(1 - \frac{\eta}{\sqrt{\eta^2-\zeta^2}}\right) \\ &= \sqrt{\eta^2-\zeta^2} - \eta = -\frac{\zeta^2}{\eta + \sqrt{\eta^2+\zeta^2}}. \end{aligned}$$

With α as above and $\beta = \sqrt{\eta^2-\zeta^2} - \eta$ we then get

$$\begin{aligned} H &= \sqrt{\eta^2-\zeta^2}\left(C_+^* C_+ + C_-^* C_-\right) - \frac{\zeta^2}{\eta + \sqrt{\eta^2+\zeta^2}} \\ &= \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta \end{aligned}$$

□